

# ON ENDOMORPHISMS OF THE CUNTZ ALGEBRA WHICH PRESERVE THE CANONICAL UHF-SUBALGEBRA, II

TOMOHIRO HAYASHI, JEONG HEE HONG, AND WOJCIECH SZYMAŃSKI

**ABSTRACT.** It was shown recently by Conti, Rørdam and Szymański that there exist endomorphisms  $\lambda_u$  of the Cuntz algebra  $\mathcal{O}_n$  such that  $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$  but  $u \notin \mathcal{F}_n$ , and a question was raised if for such a  $u$  there must always exist a unitary  $v \in \mathcal{F}_n$  with  $\lambda_u|_{\mathcal{F}_n} = \lambda_v|_{\mathcal{F}_n}$ . In the present paper, we answer this question to the negative. To this end, we analyze the structure of such endomorphisms  $\lambda_u$  for which the relative commutant  $\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n$  is finite dimensional.

## 1. INTRODUCTION AND PRELIMINARIES

This paper is devoted to continuation of the line of investigation of exotic endomorphisms of the Cuntz algebras initiated in [4]. Our main result is solution of a question raised therein, see below for details. Our strategy is based on a detailed analysis of such endomorphisms  $\lambda_u$  of  $\mathcal{O}_n$  that globally preserve the core UHF subalgebra  $\mathcal{F}_n$  and have finite dimensional relative commutant  $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ , and builds on the earlier results in this direction obtained in [10].

The Cuntz algebra  $\mathcal{O}_n$ ,  $n \geq 2$ , is the  $C^*$ -algebra generated by isometries  $S_1, \dots, S_n$  satisfying  $\sum_{i=1}^n S_i S_i^* = 1$ . It is a purely infinite, simple  $C^*$ -algebra, independent of the choice of generating isometries, [7]. We denote by  $W_n^k$  the set of  $k$ -tuples  $\mu = (\mu_1, \dots, \mu_k)$  with  $\mu_m \in \{1, \dots, n\}$ , and by  $W_n$  the union  $\cup_{k=0}^\infty W_n^k$ , where  $W_n^0 = \{0\}$ . If  $\mu \in W_n^k$  then  $|\mu| = k$  is the length of  $\mu$ . If  $\mu = (\mu_1, \dots, \mu_k) \in W_n$  then  $S_\mu = S_{\mu_1} \dots S_{\mu_k}$  ( $S_0 = 1$  by convention) is an isometry in  $\mathcal{O}_n$ . Every word in  $\{S_i, S_i^* \mid i = 1, \dots, n\}$  can be uniquely expressed as  $S_\mu S_\nu^*$ , for  $\mu, \nu \in W_n$  [7, Lemma 1.3].

The gauge action  $\gamma$  of the circle group  $\mathbb{T}$  on  $\mathcal{O}_n$  is defined by  $\gamma_z(S_i) = zS_i$ ,  $z \in \mathbb{T}$ . Let  $\mathcal{F}_n$  be the fixed point algebra of  $\gamma$ . Denote  $\mathcal{F}_n^{(k)} := \text{span}\{S_\mu S_\nu^* \mid \mu, \nu \in W_n^k\}$ . Then  $\mathcal{F}_n$  is generated by  $\mathcal{F}_n^{(k)}$ ,  $k = 1, 2, \dots$ , and each  $\mathcal{F}_n^{(k)}$  is isomorphic to the matrix algebra  $M_{n^k}(\mathbb{C})$ . Thus  $\mathcal{F}_n$  is isomorphic to the UHF-algebra of type  $n^\infty$ , and hence it has a

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unique tracial state  $\tau$ . There exists a faithful conditional expectation  $E : \mathcal{O}_n \rightarrow \mathcal{F}_n$ , defined by integration with respect to the Haar measure on  $\mathbb{T}$  as

$$E(x) = \int_{\mathbb{T}} \gamma_z(x) dz.$$

For each  $k \in \mathbb{Z}$  we denote by  $\mathcal{O}_n^{(k)}$  the corresponding spectral subspace for  $\gamma$  in  $\mathcal{O}_n$ ,

$$\mathcal{O}_n^{(k)} := \{x \in \mathcal{O}_n \mid \gamma_z(x) = z^k, \forall z \in \mathbb{T}\}.$$

Thus, in particular,  $\mathcal{O}_n^{(0)} = \mathcal{F}_n$ .

The  $C^*$ -subalgebra of  $\mathcal{O}_n$  generated by projections  $P_\mu := S_\mu S_\mu^*$ ,  $\mu \in W_n$ , is a MASA (maximal abelian subalgebra) in  $\mathcal{O}_n$ . We call it the diagonal and denote  $\mathcal{D}_n$ , also writing  $\mathcal{D}_n^k$  for  $\mathcal{D}_n \cap \mathcal{F}_n^{(k)}$ .

The canonical shift endomorphism  $\varphi : \mathcal{O}_n \rightarrow \mathcal{O}_n$  is defined by

$$\varphi(x) = \sum_{i=1}^n S_i x S_i^*.$$

It is easy to see that  $S_i x = \varphi(x) S_i$  and  $x S_i^* = S_i^* \varphi(x)$  for all  $x \in \mathcal{O}_n$ .

As shown by Cuntz in [8], there exists a bijective correspondence between unitaries in  $\mathcal{O}_n$  (whose collection is denoted  $\mathcal{U}(\mathcal{O}_n)$ ) and unital  $*$ -endomorphisms of  $\mathcal{O}_n$ , determined by

$$\lambda_u(S_i) = u S_i, \quad i = 1, \dots, n.$$

We have  $\text{Ad}(u) = \lambda_{u\varphi(u^*)}$  for all  $u \in \mathcal{U}(\mathcal{O}_n)$ . If  $u \in \mathcal{U}(\mathcal{O}_n)$  then for each positive integer  $k$  we denote

$$(1) \quad u_k = u\varphi(u) \cdots \varphi^{k-1}(u).$$

Here  $\varphi^0 = \text{id}$ , and we agree that  $u_k^*$  stands for  $(u_k)^*$ . If  $\alpha$  and  $\beta$  are multi-indices of length  $k$  and  $m$ , respectively, then  $\lambda_u(S_\alpha S_\beta^*) = u_k S_\alpha S_\beta^* u_m^*$ .

The Cuntz correspondence between unitaries and endomorphisms of  $\mathcal{O}_n$  provides a very efficient tool for investigations of the latter. In this note, we continue the study (by several authors) of those unital endomorphisms which globally preserve the UHF-subalgebra  $\mathcal{F}_n$ . For example, such endomorphisms were analyzed from the point of view of the Jones-Kosaki-Watatani index theory in [12] and [3], and in connection with Hopf algebra actions in [9] and [13]. More recently, interesting combinatorial approaches to the study of permutative endomorphisms of this type have been found (e.g. see [6], [2], and a survey article [1]).

It was observed by Cuntz in his groundbreaking paper [8] that an *automorphism*  $\lambda_u$  globally preserves  $\mathcal{F}_n$  if and only if  $u \in \mathcal{F}_n$ . The situation is more complex with *proper endomorphisms*. Clearly,  $u \in \mathcal{F}_n$  implies  $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$ , [8], but the question if the converse is true remained open until very recently. Indeed, it was shown in [4] that

there exist unitaries  $u$  in  $\mathcal{O}_n \setminus \mathcal{F}_n$  such that  $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$ . All such examples found therein were of the form  $u = wv$  with  $w \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$  and  $v \in \mathcal{F}_n$ . In such a case, we also have  $\lambda_u(x) = \lambda_v(x)$  for all  $x \in \mathcal{F}_n$ . Thus a natural question arises if such a factorization of  $u$  is always possible whenever  $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$  (cf. [1, Problem 5.3]).

Some progress towards answering this question has been made recently in [10] and [11]. The main purpose of the present paper is to develop definite methods for analyzing endomorphisms  $\lambda_u$  of  $\mathcal{O}_n$  satisfying  $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$  and an additional condition that the relative commutant  $\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n$  be finite dimensional. In particular, we give a verifiable criterion for determining if the aforementioned decomposition is possible, Corollary 3.4. Based on this criterion, in Section 3 we give an explicit example of a unitary  $u \in \mathcal{O}_2$  such that  $\lambda_u(\mathcal{F}_2) \subseteq \mathcal{F}_2$  and  $\dim \lambda_u(\mathcal{F}_2)' \cap \mathcal{F}_2 < \infty$  but there is no unitary  $v \in \mathcal{F}_2$  such that  $\lambda_u|_{\mathcal{F}_2} = \lambda_v|_{\mathcal{F}_2}$ , see Example 3.6. In this way, we answer to the negative the question raised in [4] and [1].

## 2. THE RELATIVE COMMUTANTS

We begin by recording for future references a few simple facts, essentially contained in [4] and [10].

**Proposition 2.1.** *Let  $u \in \mathcal{U}(\mathcal{O}_n)$ . Then the following conditions are equivalent.*

- (1)  $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$ ,
- (2)  $\lambda_{\gamma_z(u)}|_{\mathcal{F}_n} = \lambda_u|_{\mathcal{F}_n}$  for all  $z \in \mathbb{T}$ ,
- (3)  $u\gamma_z(u^*) \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$  for all  $z \in \mathbb{T}$ .

*Proof.* Clearly,  $\gamma_z \lambda_u \gamma_z^{-1} = \lambda_{\gamma_z(u)}$  for all  $z \in \mathbb{T}$ . Thus condition (2) above is equivalent to  $\gamma_z \lambda_u|_{\mathcal{F}_n} = \lambda_u|_{\mathcal{F}_n}$  for all  $z \in \mathbb{T}$ . Obviously, this holds if and only if  $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$ . That is, (1) is equivalent to (2).

It is an immediate consequence of Proposition 2.1 and Proposition 4.7 from [4] that  $\lambda_u|_{\mathcal{F}_n} = \lambda_v|_{\mathcal{F}_n}$  if and only if  $vu^* \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ . This gives (2) is equivalent to (3).  $\square$

**Proposition 2.2.** *If  $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$  and  $\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n = \mathbb{C}1$ , then  $u \in \mathcal{F}_n$ .*

*Proof.* If  $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$  and  $\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n = \mathbb{C}1$ , then  $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n = \mathbb{C}1$  as well, [10, Theorem 1.1]. As shown in [4], this implies that  $u \in \mathcal{F}_n$ .  $\square$

**Proposition 2.3.** *Let  $u$  be a unitary in  $\mathcal{O}_n$ . Then  $u = wv$  for some  $w \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$  and a unitary  $v \in \mathcal{F}_n$  if and only if there exists a unitary  $y \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$  such that  $u\gamma_z(u^*) = y\gamma_z(y^*)$  for all  $z \in \mathbb{T}$ .*

*Proof.* If  $u = wv$  for some  $w \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$  and  $v \in \mathcal{U}(\mathcal{F}_n)$ , then  $u\gamma_z(u^*) = w\gamma_z(w^*)$ , and it suffices to put  $y = w$ .

Conversely, if there exists a unitary  $y \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$  such that  $u\gamma_z(u^*) = y\gamma_z(y^*)$  for all  $z \in \mathbb{T}$  then  $y^*u$  is fixed by all  $\gamma_z$ . Thus  $y^*u \in \mathcal{F}_n$  and it suffices to put  $w = y$  and  $v = y^*u$ .  $\square$

From now on, we make a **standing assumption** that  $u \in \mathcal{U}(\mathcal{O}_n)$  is such that

$$(2) \quad \lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n \text{ and } \dim \lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n < \infty.$$

As shown in [10], assumption (2) above entails a number of important consequences, which we summarize as follows.

- We also have  $\dim \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n < \infty$ .
- There exists a unitary group  $\{u_z\}_{z \in \mathbb{T}}$  in the center of  $\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n$  such that  $\text{Ad} u_z(x) = \gamma_z(x)$  for all  $x \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ .
- Minimal projections in  $\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n$  are minimal in  $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$  as well. Thus  $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$  contains a MASA consisting of projections in  $\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n$ .

The proof of the following theorem is modelled after that of [10, Lemma 1.11].

**Theorem 2.4.** *Let  $u \in \mathcal{U}(\mathcal{O}_n)$  be such that  $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$  and  $\dim \lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n < \infty$ . Then there exist unitaries  $w \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$  and  $v \in \mathcal{O}_n$ , and a unitary group  $\{v_z\}_{z \in \mathbb{T}} \subseteq \lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n$  satisfying  $u = wv$  and  $\gamma_z(v) = v_z v$  for all  $z \in \mathbb{T}$ .*

*Proof.* At first we note that  $u\gamma_z(u^*)u_z$  is a unitary group in  $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ . Indeed,

$$\begin{aligned} (u\gamma_{z_1}(u^*)u_{z_1})(u\gamma_{z_2}(u^*)u_{z_2}) &= u\gamma_{z_1}(u^*)(\text{Ad } u_{z_1})(u)u_{z_1}\gamma_{z_2}(u^*)u_{z_2} \\ &= u\gamma_{z_1}(u^*)\gamma_{z_1}(u)\gamma_{z_1}(\gamma_{z_2}(u^*))u_{z_1}u_{z_2} = u\gamma_{z_1 z_2}(u^*)u_{z_1 z_2}. \end{aligned}$$

Since  $\dim \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n < \infty$ , this unitary group may be diagonalized. On the other hand,  $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$  contains a MASA composed of projections in  $\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n$ . Thus, there exists a unitary  $w \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$  such that  $y_z := w^*(u\gamma_z(u^*)u_z)w$  is a unitary group in  $\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n$ . Since each  $u_z$  is in the center of  $\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n$ , the unitary groups  $\{y_z\}_{z \in \mathbb{T}}$  and  $\{u_z\}_{z \in \mathbb{T}}$  commute.

Set  $v_z := u_z y_z^*$ ,  $z \in \mathbb{T}$ , and  $v := w^*u$ . Then  $v_z$  is a unitary group in  $\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n$  and

$$\gamma_z(v) = \gamma_z(w^*u) = u_z(u_z^* \gamma_z(w^*u) u_z^*) w^*u = u_z y_z^* w^*u = v_z w^*u = v_z v.$$

for all  $z \in \mathbb{T}$ . This completes the proof.  $\square$

We keep the notation from Theorem 2.4, assuming that unitaries  $w$ ,  $v$  and  $v_z$  have the properties described therein. Thus, in particular,  $\lambda_u|_{\mathcal{F}_n} = \lambda_v|_{\mathcal{F}_n}$  by [4, Proposition 2.1]. Consequently,  $\text{Ad } v \circ \varphi$  is an automorphism of  $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ , by [4, Proposition 2.3 and Lemma 2.4].

**Lemma 2.5.** *With unitaries  $u$ ,  $v$ ,  $v_z$  and  $u_z$  as above, put*

$$X_z := (\text{Ad } v \circ \varphi)(u_z)u_z^*v_z.$$

Then  $\{X_z\}_{z \in \mathbb{T}}$  is a unitary group in the center of  $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ , and we have

$$\gamma_z(v) = X_z u_z (\text{Ad } v \circ \varphi)(u_z^*) v.$$

*Proof.* For each  $x \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ , we see that

$$\begin{aligned} u_z (\text{Ad } v \circ \varphi)(x) u_z^* &= \gamma_z(v \varphi(x) v^*) = \gamma_z(v) \gamma_z(\varphi(x)) \gamma_z(v^*) = \gamma_z(v) \varphi(\gamma_z(x)) \gamma_z(v)^* \\ &= v_z v \varphi(u_z x u_z^*) v^* v_z^* = v_z v \varphi(u_z) v^* v \varphi(x) v^* v \varphi(u_z^*) v^* v_z^* \\ &= v_z \text{Ad}(v \varphi(u_z) v^*) ((\text{Ad } v \circ \varphi)(x)) v_z^*. \end{aligned}$$

Hence, we have

$$\text{Ad}(v_z^* u_z) ((\text{Ad } v \circ \varphi)(x)) = \text{Ad}((\text{Ad } v \circ \varphi)(u_z)) ((\text{Ad } v \circ \varphi)(x)).$$

Since  $\text{Ad } v \circ \varphi$  is an automorphism of  $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ , this shows that

$$(3) \quad \text{Ad}(v_z^* u_z) = \text{Ad}((\text{Ad } v \circ \varphi)(u_z)) \text{ on } \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n.$$

Consequently,  $X_z$  belongs to the center of  $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ .

Now,  $\{u_z\}_{z \in \mathbb{T}}$  and  $\{v_z\}_{z \in \mathbb{T}}$  are commuting unitary groups, and both commute with  $X_z$ , by the above argument. Therefore the unitary group  $(\text{Ad } v \circ \varphi)(u_z) = X_z u_z v_z^*$  commutes with both of them. Consequently,  $X_z$  being a product of three mutually commuting unitary groups itself is a unitary group.

The final claim of the lemma now follows from the fact that  $\gamma_z(v) = v_z v$ .  $\square$

Before proceeding further, we introduce the following notation. For  $x \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$  and  $k \in \mathbb{N}$ , we set

$$(4) \quad x^{(k)} := x (\text{Ad } v \circ \varphi)(x) (\text{Ad } v \circ \varphi)^2(x) \cdots (\text{Ad } v \circ \varphi)^{k-1}(x).$$

**Lemma 2.6.** *With unitaries  $u$ ,  $v$ ,  $v_z$  and  $u_z$  as above, and  $x \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ , for all  $g \in \mathcal{U}(\mathcal{O}_n)$ ,  $z \in \mathbb{T}$  and  $k \in \mathbb{N}$  we have the following identities.*

- (i)  $\gamma_z(v_k) = v_z^{(k)} v_k$ ,
- (ii)  $(\text{Ad } g \circ \varphi)^k(x) = g_k \varphi^k(x) g_k^*$ ,
- (iii)  $v_z^{(k)} = X_z^{(k)} u_z (\text{Ad } v \circ \varphi)^k(u_z^*)$ ,
- (iv)  $(gv)_k = g^{(k)} v_k$ .

*Proof.* In all three cases, we proceed by induction on  $k$ .

Ad (i). Case  $k = 1$  is the identity  $\gamma_z(v_1) = \gamma_z(v) = v_z v = v_z^{(1)} v_1$  from Theorem 2.4. For the inductive step, we calculate

$$\gamma_z(v_{k+1}) = \gamma_z(v_k \varphi^k(v)) = \gamma_z(v_k) \varphi^k(\gamma_z(v)) = v_z^{(k)} v_k \varphi^k(v_z^* v_k \varphi^k(v)) = v_z^{(k+1)} v_{k+1}.$$

In this calculation we used identity (ii) of the present lemma, whose proof does not depend on (i).

Ad (ii). Case  $k = 1$  is clear. For the inductive step, we have

$$(\text{Ad } g \circ \varphi)^{k+1} = (\text{Ad } g \circ \varphi)(g_k \varphi^k(x) g_k^*) = g \varphi(g_k) \varphi^{k+1}(x) \varphi(g_k^*) g^* = g_{k+1} \varphi^{k+1}(x) g_{k+1}^*.$$

Ad (iii). Case  $k = 1$  is clear. For the inductive step, we see that

$$\begin{aligned} v_z^{(k+1)} &= v_z^{(k)} (\text{Ad } v \circ \varphi)^k(v_z) = X_z^{(k)} u_z (\text{Ad } v \circ \varphi)^k(u_z^*) (\text{Ad } v \circ \varphi)^k(v_z) \\ &= X_z^{(k)} u_z (\text{Ad } v \circ \varphi)^k(u_z^* v_z) = X_z^{(k)} u_z (\text{Ad } v \circ \varphi)^k(X_z (\text{Ad } v \circ \varphi)(u_z^*)) \\ &= X_z^{(k)} u_z (\text{Ad } v \circ \varphi)^k(X_z) (\text{Ad } v \circ \varphi)^{k+1}(u_z^*) = X_z^{(k+1)} u_z (\text{Ad } v \circ \varphi)^{k+1}(u_z^*). \end{aligned}$$

Ad (iv). Case  $k = 1$  is clear. For the inductive step, we calculate using part (ii) above,

$$(gv)_{k+1} = (gv)_k \varphi^k(gv) = g^{(k)} v_k \varphi^k(gv) = g^{(k)} (v_k \varphi^k(g) v_k^*) v_k \varphi^k(v) = g^{(k+1)} v_{k+1},$$

and this completes the proof.  $\square$

The following lemma provides a key step in the proof of our second main result, Theorem 2.8, below. We continue keeping the notation of Theorem 2.4. Here we remark that since  $v = w^* u$ ,  $w \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ , and  $\text{Ad } u \circ \varphi$  is an automorphism of  $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ , we see that  $\text{Ad } v \circ \varphi = \text{Ad } w^* \circ (\text{Ad } u \circ \varphi)$  is an automorphism of  $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$  as well. We also note that for each positive integer  $k$ ,  $\{X_z^{(k)}\}_{z \in \mathbb{T}}$  is a unitary group in the center of  $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ .

**Lemma 2.7.** *With unitaries  $u$ ,  $v$ ,  $v_z$  and  $u_z$  as above, there exist a positive integer  $k$  and a unitary  $U \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$  such that*

$$(\text{Ad } v \circ \varphi)^k(x) = \text{Ad } U(x) \quad \text{for all } x \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n.$$

*Then  $X_z^{(k)} = 1$ . Furthermore, for such  $U$  and  $k$ , we have  $U^* v_k \in \mathcal{F}_n$ .*

*Proof.* Since  $\text{Ad } v \circ \varphi$  is an automorphism of a finite dimensional  $C^*$ -algebra  $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ , its restriction to the center has finite order. Thus there exists a positive integer  $k$  and a unitary  $U \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$  such that  $(\text{Ad } v \circ \varphi)^k = \text{Ad } U$  on  $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ . We claim that  $U^* v_k \in \mathcal{F}_n$ .

Indeed, by Lemma 2.6, for all  $z \in \mathbb{T}$  we have

$$\gamma_z(v_k) = v_z^{(k)} v_k = X_z^{(k)} u_z (\text{Ad } v \circ \varphi)^k(u_z^*) v_k = X_z^{(k)} u_z U u_z^* U^* v_k = X_z^{(k)} \gamma_z(U) U^* v_k,$$

and this yields

$$(5) \quad \gamma_z(U^* v_k) = X_z^{(k)} U^* v_k.$$

Since  $\{X_z^{(k)}\}_{z \in \mathbb{T}}$  is a unitary group in the center of  $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ , there exists a partition of unity  $1 = \sum_i p_i$  in  $\mathcal{Z}(\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n)$  and integers  $k_i$  such that

$$X_z^{(k)} = \sum_i z^{k_i} p_i.$$

We have  $(\text{Ad } u \circ \varphi)^k(p_i) = Up_iU^* = p_i$  for all  $i$ . Combining this with part (ii) of Lemma 2.6, we get

$$(6) \quad v_k^* p_i v_k = \varphi^k(p_i).$$

We want to show that  $k_i = 0$  for all  $i$ . Suppose for a moment this is not the case and let  $k_i > 0$  for some  $i$ . We set  $K := p_i U^* v_k (S_1^*)^{k_i}$ . Since  $p_i$  being in  $\mathcal{Z}(\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n)$  belongs to  $\mathcal{F}_n$  as well, it follows from identity (5) above that

$$\gamma_z(K) = \gamma_z(p_i U^* v_k (S_1^*)^{k_i}) = p_i X_z^{(k)} U^* v_k \gamma_z((S_1^*)^{k_i}) = z^{k_i} p_i U^* v_k (z^{-k_i} (S_1^*)^{k_i}) = K.$$

Hence  $K$  belongs to  $\mathcal{F}_n$ . We have  $KK^* = p_i$ . On the other hand, using identity (6) we get

$$K^* K = S_1^{k_i} v_k^* p_i v_k (S_1^*)^{k_i} = S_1^{k_i} \varphi^k(p_i) (S_1^*)^{k_i} = \varphi^{k+k_i}(p_i) S_1^{k_i} (S_1^*)^{k_i}.$$

It easily follows that  $\tau(KK^*) > \tau(K^*K)$ , which is a contradiction. A similar argument applies in the case  $k_i < 0$ . Hence  $k_i = 0$  for all  $i$  and thus  $X_z^{(k)} = 1$ . Now, identity (5) implies that  $U^* v_k$  is fixed by the gauge action and hence belongs to  $\mathcal{F}_n$ .  $\square$

Now, we are ready to prove the second main result of this paper.

**Theorem 2.8.** *Let  $u \in \mathcal{U}(\mathcal{O}_n)$  be such that  $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$  and  $\dim \lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n < \infty$ . Then there exist a positive integer  $k$  and unitaries  $W \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$  and  $V \in \mathcal{F}_n$  such that  $u_k = WV$ .*

*Proof.* By Theorem 2.4 and Lemma 2.7, there exist unitaries  $w, U \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ , a unitary group  $\{v_z\}_{z \in \mathbb{T}}$  in  $\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n$  and a positive integer  $k$  satisfying  $u = wv$ ,  $\gamma_z(v) = v_z v$ ,  $U^* v_k \in \mathcal{F}_n$ . By part (iv) of Lemma 2.6, we have  $w^{(k)} v_k = u_k$ . Thus to complete the proof, it suffices to put  $W := w^{(k)} U$  and  $V := U^* v_k$ .  $\square$

It was observed in [4] (just above Remark 4.4) that if  $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$  and  $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n = \mathbb{C}1$  then  $u \in \mathcal{F}_n$ . The following corollary gives a sharp strengthening of that result.

**Corollary 2.9.** *Let  $u$  be a unitary in  $\mathcal{O}_n$ . If  $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$ ,  $\dim \lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n < \infty$  and the automorphism  $\text{Ad } u \circ \varphi$  of  $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$  is inner, then there exist a unitary  $w \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$  and a unitary  $v \in \mathcal{F}_n$  such that  $u = wv$ , and hence also  $\lambda_u|_{\mathcal{F}_n} = \lambda_v|_{\mathcal{F}_n}$ . In particular, this is the case whenever  $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$  is a factor.*

**Remark 2.10.** The assumption in Corollary 2.9 above that the automorphism  $\text{Ad } u \circ \varphi$  of  $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$  be inner, is equivalent to demanding existence of a unitary  $g$  in the relative commutant  $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$  such that

$$\lambda_{gu}(\mathcal{F}_n)' \cap \mathcal{O}_n = \lambda_{gu}(\mathcal{O}_n)' \cap \mathcal{O}_n.$$

Indeed, if  $\text{Ad } u \circ \varphi$  is inner then  $\text{Ad } gu \circ \varphi = \text{id}$  for a suitable unitary  $g$  in  $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ . Hence  $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n = \lambda_{gu}(\mathcal{F}_n)' \cap \mathcal{O}_n = \lambda_{gu}(\mathcal{O}_n)' \cap \mathcal{O}_n$ . Conversely, if  $\lambda_{gu}(\mathcal{F}_n)' \cap \mathcal{O}_n =$



$\lambda_{gu}(\mathcal{O}_n)' \cap \mathcal{O}_n$  then  $\text{Ad } gu \circ \varphi = \text{id}$  on  $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n = \lambda_{gu}(\mathcal{F}_n)' \cap \mathcal{O}_n$ , and hence  $\text{Ad } u \circ \varphi$  is inner.  $\square$

**Remark 2.11.** We remark that the implication in Corollary 2.9 above cannot be reversed. In fact, there exist unitaries  $u \in \mathcal{F}_n$  such that  $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$  is finite dimensional and the automorphism  $\text{Ad } u \circ \varphi$  is outer on  $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ . For example, take

$$u = S_{22}S_{11}^* + S_{12}S_{22}^* + S_{11}S_{12}^* + P_{21},$$

a permutative unitary in  $\mathcal{F}_2$ . Then  $\text{Ad } u \circ \varphi$  is outer on  $\lambda_u(\mathcal{F}_2)' \cap \mathcal{O}_2$ . For otherwise let  $h$  be a unitary in  $\lambda_u(\mathcal{F}_2)' \cap \mathcal{O}_2$  such that  $\text{Ad } u \circ \varphi = \text{Ad } h$  on  $\lambda_u(\mathcal{F}_2)' \cap \mathcal{O}_2$ . Then  $\text{Ad } u \circ \varphi(h) = h$  and thus  $h \in \lambda_u(\mathcal{O}_2)' \cap \mathcal{O}_2$ . But it can be shown that  $\lambda_u$  is irreducible on  $\mathcal{O}_2$  (e.g., see [5], where this endomorphism is denoted  $\rho_{142}$ ), and hence  $\lambda_u(\mathcal{O}_2)' \cap \mathcal{O}_2 = \mathbb{C}1$ . Thus  $h$  is a scalar and consequently  $\text{Ad } u \circ \varphi$  is identity on  $\lambda_u(\mathcal{F}_2)' \cap \mathcal{O}_2$ . This however is not the case, since one can calculate directly that  $\text{Ad } u \circ \varphi$  permutes  $P_1$  and  $P_2$ , and both these projections are in  $\lambda_u(\mathcal{F}_2)' \cap \mathcal{O}_2$ .  $\square$

We want to elaborate a little bit the statement of Theorem 2.8 above. We continue keeping our standing assumption (2).

**Lemma 2.12.** *Let  $\alpha$  be an automorphism of  $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$  and let  $k \in \mathbb{N}$  be such that  $\alpha^k$  acts trivially on  $\mathcal{Z}(\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n)$ . Then there exists a MASA  $D$  of  $\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n$  and a unitary  $g$  in  $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$  such that*

- (i)  $(\text{Ad } g \circ \alpha)^k = \text{id}$ , and
- (ii)  $(\text{Ad } g \circ \alpha)(D) = D$ .

*Proof.* Automorphism  $\alpha$  permutes the finitely many minimal central projections of  $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ . Write this permutation as a product of disjoint cycles. Clearly, it suffices to prove the lemma for each cycle separately. Thus we may simply assume that  $\alpha$  acts transitively on minimal projections  $p_1, p_2, \dots, p_l$  in  $\mathcal{Z}(\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n)$ , so that  $\alpha(p_i) = p_{i+1}$ , with  $p_{l+1} = p_1$ . Let  $\{e_{r,s}^{(i)}\}$  be matrix units of the full matrix algebra  $p_i(\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n)$ , such that all  $e_{r,r}^{(i)}$  are in  $\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n$ . Then  $D := \text{span}\{e_{r,r}^{(i)}\}$  is a MASA in  $\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n$ . Since  $p_i(\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n) \cong p_{i+1}(\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n)$ , we can find a unitary  $g_i \in p_{i+1}(\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n)$  such that  $(\text{Ad } g_i \circ \alpha)(e_{r,s}^{(i)}) = e_{r,s}^{(i+1)}$ . Setting  $g := \sum_{i=1}^l g_i$  we obtain the desired result.  $\square$

**Lemma 2.13.** *Let  $u \in \mathcal{U}(\mathcal{O}_n)$  be such that  $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$  and  $\dim \lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n < \infty$ . Then there exist a positive integer  $k$ , a unitary  $g \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ , and a unitary group  $\{d_z\}_{z \in \mathbb{T}} \subseteq \lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n$  such that  $(gv)_k \in \mathcal{F}_n$  and  $\gamma_z(gv) = d_z gv$ .*

*Proof.* Put  $\alpha := \text{Ad } v \circ \varphi$ , and let  $g$  and  $k$  be as in Lemma 2.12. Then we have

$$(\text{Ad } gv \circ \varphi)^k = \text{id},$$



and thus

$$\text{Ad } v_k \circ \varphi^k = (\text{Ad } v \circ \varphi)^k = \text{Ad}(g^{(k)})^*$$

by parts (ii) and (iv) of Lemma 2.6. Then arguing as in the proof of Lemma 2.7 (with  $g^{(k)*}$  playing the role of  $U$ ), we get

$$(gv)_k = g^{(k)}v_k \in \mathcal{F}_n.$$

Now, let  $D$  be a MASA as in Lemma 2.12. For all  $x \in D$  and  $z \in \mathbb{T}$ , we see that

$$\begin{aligned} gv\varphi(x)v^*g^* &= \gamma_z(gv\varphi(x)v^*g^*) = \gamma_z(g)v_zv\varphi(x)v^*v_z^*\gamma_z(g^*) \\ &= (\gamma_z(g)v_zg^*)(gv\varphi(x)v^*g^*)(\gamma_z(g)v_zg)^*, \end{aligned}$$

which implies that  $\gamma_z(g)v_zg^*$  is in the commutant of MASA  $D$ , and hence in  $D$  itself. Set  $d_z = \gamma_z(g)v_zg^*$ , a unitary in  $D$ . Now,  $d_z = u_zgu_z^*v_zg^*$  implies  $u_z^*d_z = g(u_z^*v_z)g^*$ . Since  $\{u_z\}_{z \in \mathbb{T}}$  and  $\{v_z\}_{z \in \mathbb{T}}$  are commuting unitary groups, so is  $\{u_z^*d_z\}_{z \in \mathbb{T}}$ , and consequently also is  $\{d_z\}_{z \in \mathbb{T}}$ . Finally, we see that  $\gamma_z(gv) = \gamma_z(g)v_zv = d_zgv$ .  $\square$

Now, we are ready to prove the following result.

**Theorem 2.14.** *Let  $u \in \mathcal{U}(\mathcal{O}_n)$ . If  $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$  and  $\dim \lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n < \infty$ , then there exists a unitary  $W \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$  satisfying the following.*

- (i) *There exists a unitary group  $\{d_z\}_{z \in \mathbb{T}} \subseteq \lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n$  such that  $\gamma_z(Wu) = d_zWu$  for all  $z \in \mathbb{T}$ .*
- (ii) *There exists a positive integer  $k$  such that  $(Wu)_k \in \mathcal{F}_n$ .*

*Proof.* Let  $u = wv$  be a factorization as in Theorem 2.4, and let  $k \in \mathbb{N}$  and  $g \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$  be as in Lemma 2.13 above. Then setting  $W := gw^*$  gives the claim.  $\square$

### 3. THE CRITERION AND EXAMPLES

In this section, we give a dynamic characterization of those unitaries  $u \in \mathcal{O}_n$  satisfying our standing assumptions which either belong to  $\mathcal{F}_n$  (Theorem 3.2) or admit a unitary  $v \in \mathcal{F}_n$  such that  $\lambda_u|_{\mathcal{F}_n} = \lambda_v|_{\mathcal{F}_n}$  (Corollary 3.4). Before proving these results, we still need one technical lemma about the structure of the relative commutants. We keep our standing assumptions (2).

**Lemma 3.1.** *There exist a unitary group  $\{q_z\}_{z \in \mathbb{T}}$  in  $\mathcal{Z}(\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n)$  such that*

$$X_z = q_z(\text{Ad } v \circ \varphi)(q_z^*).$$

*Proof.* Since  $\text{Ad } v \circ \varphi$  restricts to an automorphism of  $\mathcal{Z}(\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n)$ , there exist minimal projections  $p_i^{(j)}$ ,  $j = 1, \dots, N$ ,  $i = 1, \dots, n_j$ , in  $\mathcal{Z}(\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n)$  such that

$$\mathcal{Z}(\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n) = \bigoplus_{j=1}^N \bigoplus_{i=1}^{n_j} p_i^{(j)}$$

and

$$(\text{Ad } v \circ \varphi)(p_i^{(j)}) = p_{i+1}^{(j)} \text{ for } i < n_j, \text{ and } (\text{Ad } v \circ \varphi)(p_{n_i}^{(j)}) = p_1^{(j)}.$$

Then  $X_z$  from Lemma 2.5 can be written as

$$X_z = \sum_{j=1}^N \sum_{i=1}^{n_j} z^{m_i^{(j)}} p_i^{(j)},$$

for some  $m_i^{(j)} \in \mathbb{N}$ . Now, let  $k \in \mathbb{N}$  be such that  $\text{Ad } v \circ \varphi$  is an inner automorphism of  $\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ . Then

$$X_z^{(k)} = X_z(\text{Ad } v \circ \varphi)(X_z)(\text{Ad } v \circ \varphi)^2(X_z) \dots (\text{Ad } v \circ \varphi)^{k-1}(X_z) = 1$$

by Lemma 2.7. Since each  $n_j$  divides  $k$ , this implies that

$$\sum_{i=1}^{n_j} m_i^{(j)} = 0$$

for each  $j = 1, \dots, N$ . Now, we want to define  $q_z$  as follows,

$$q_z = \sum_{j=1}^N \sum_{i=1}^{n_j} z^{r_i^{(j)}} p_i^{(j)},$$

for suitable chosen integers  $r_i^{(j)}$ , so that  $X_z = q_z(\text{Ad } v \circ \varphi)(q_z^*)$ . To this end, it suffices to put

$$\begin{aligned} r_1^{(j)} &:= 0, \quad j = 1, \dots, N, \\ r_k^{(j)} &:= \sum_{r=2}^k m_r^{(j)}, \quad j = 1, \dots, N, \quad k = 2, \dots, n_j. \end{aligned}$$

□

**Theorem 3.2.** *Let  $u \in \mathcal{U}(\mathcal{O}_n)$  be such that  $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$  and  $\dim \lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n < \infty$ . Put  $\alpha := \text{Ad } u \circ \varphi$ . If  $\alpha$  satisfies the following two conditions:*

- (i)  $\alpha(\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n) = \lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n$ , and
- (ii)  $\alpha|_{\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n}$  preserves the  $\tau$ -trace,

then  $u \in \mathcal{F}_n$ .

*Proof.* At first, we observe that there exists a unitary group  $\{u'_z\}_{z \in \mathbb{T}}$  in  $\mathcal{Z}(\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n)$  such that  $\text{Ad } u'_z(x) = \gamma_z(x)$  for all  $x \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$  and  $\gamma_z(u) = u'_z \alpha(u'_z^*) u$ . Indeed, it suffices to put  $u'_z := q_z u_z$ , with  $q_z$  as in Lemma 3.1 above. Then  $\alpha(u'_z) \in \mathcal{Z}(\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n)$  by condition (i) of the theorem, and hence  $\{u'_z \alpha(u'_z^*)\}_{z \in \mathbb{T}}$  is a unitary group. Thus,  $u'_z \alpha(u'_z^*) = \sum z^{k_j} p_j$  for some integers  $k_j$  and a partition of unity by projections  $p_j$  from  $\mathcal{Z}(\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n)$ .

Now, we claim that  $p_j = 0$  whenever  $k_j \neq 0$ . To this end, suppose first that  $k_j > 0$  for some index  $j$ , and put  $R := p_{k_j} u(S_1^*)^{k_j}$ . We have  $\gamma_z(R) = R$  for all

$z \in \mathbb{T}$ , and thus  $R \in \mathcal{F}_n$ . However, an easy calculation shows that  $RR^* = p_{k_j}$  and  $R^*R = \varphi^{k+1}(\alpha^{-1}(p_{k_j}))S_1^k(S_1^*)^k$ . In view of condition (ii) of the theorem, this would imply  $\tau(RR^*) \neq \tau(R^*R)$  if  $p_j \neq 0$ , a contradiction. Therefore  $p_j = 0$  for all  $k_j > 0$ . A similar argument shows that  $p_j = 0$  if  $k_j < 0$ .

Consequently,  $u'_z \alpha(u'_z)^* = 1$ . But this gives  $\gamma_z(u) = u$  for all  $z \in \mathbb{T}$ . Hence  $u \in \mathcal{F}_n$  and the theorem is proved.  $\square$

We note that Theorem 3.2 gives a necessary and sufficient condition for  $u \in \mathcal{F}_n$ , since the reverse implication is trivial. Likewise, Corollary 3.3 below, gives a necessary and sufficient condition for  $u_k \in \mathcal{F}_n$ .

**Corollary 3.3.** *Let  $u \in \mathcal{U}(\mathcal{O}_n)$  be such that  $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$  and  $\dim \lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n < \infty$ . Put  $\alpha := (\text{Ad } u \circ \varphi)^k$ , for some positive integer  $k$ . If  $\alpha$  satisfies the following two conditions:*

- (i)  $\alpha(\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n) = \lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n$ , and
- (ii)  $\alpha|_{\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n}$  preserves the  $\tau$ -trace,

then  $u_k \in \mathcal{F}_n$ .

Now, we are ready to give the following decomposability criterion.

**Corollary 3.4.** *Let  $u \in \mathcal{U}(\mathcal{O}_n)$  be such that  $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$  and  $\dim \lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n < \infty$ . Put  $\alpha := \text{Ad } u \circ \varphi$ . Then the following two conditions are equivalent:*

- (1) *There exist unitaries  $w \in \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$  and  $v \in \mathcal{F}_n$  such that  $u = wv$ .*
- (2) *For each minimal projection  $p \in \mathcal{Z}(\lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n)$  there exists a  $\tau$ -preserving isomorphism*

$$p(\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n) \cong \alpha(p)(\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n).$$

Now, we show how to construct examples of endomorphisms  $\lambda_u$  of  $\mathcal{O}_n$  globally preserving the core UHF-subalgebra  $\mathcal{F}_n$  but such that no unitary  $v \in \mathcal{F}_n$  exists for which  $\lambda_u|_{\mathcal{F}_n} = \lambda_v|_{\mathcal{F}_n}$ .

To begin with, take two non-zero, orthogonal projections  $q_1, q_2$  in  $\mathcal{F}_n$  such that  $\tau(q_2)/\tau(q_1) = n^r$  for some non-zero integer  $r$ . Let  $A_1$  be a partial isometry in  $\mathcal{O}_n^{(-r)}$  with domain projection  $\varphi(q_1)$  and range projection  $q_2$ . Likewise, let  $A_2$  be a partial isometry in  $\mathcal{O}_n^{(r)}$  with domain projection  $\varphi(q_2)$  and range projection  $q_1$ . Finally, let  $A_3$  be a partial isometry in  $\mathcal{F}_n$  with domain projection  $1 - \varphi(q_1) - \varphi(q_2)$  and range projection  $1 - q_1 - q_2$ . Put  $u := A_1 + A_2 + A_3$ . Then  $u$  is a unitary in  $\mathcal{O}_n$  such that

$$(7) \quad \text{Ad } u \circ \varphi(q_1) = q_2 \quad \text{and} \quad \text{Ad } u \circ \varphi(q_2) = q_1.$$

Now,  $u\gamma_z(u^*) = z^r q_1 + z^{-r} q_2 + 1 - q_1 - q_2$  belongs to  $\text{span}\{1, q_1, q_2\}$ , and  $\text{span}\{1, q_1, q_2\} \subseteq \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$  by [4, Proposition 2.3] and (7) above. Thus  $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$  by Proposition 2.1 above.

More generally, Let  $1 = \sum q_j$  be a partition of unity by projections in  $\mathcal{O}_n$ . Let  $u$  be any unitary in  $\mathcal{O}_n$  such that  $\text{Ad } u \circ \varphi$  permutes projections  $\{q_j\}$  and for each  $j$  there is a  $k_j \in \mathbb{Z}$  such that  $q_j u \in \mathcal{O}_n^{(k_j)}$ . Then  $u \gamma_z(u^*) \in \text{span}\{q_j\} \subseteq \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$  for all  $z \in \mathbb{T}$ . This simple construction gives a large class of examples of unitaries  $u \in \mathcal{O}_n \setminus \mathcal{F}_n$  such that  $\lambda_u(\mathcal{F}_n) \subseteq \mathcal{F}_n$ . However, to verify the conditions of Corollary 3.4 one needs more detailed information on the relative commutants  $\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n \subseteq \lambda_u(\mathcal{F}_n)' \cap \mathcal{O}_n$ . Exact determination of these relative commutants is rather difficult and does not seem possible in general, despite the identity from [4, Proposition 2.3]. However, it is quite doable in concrete cases.

Now, we illustrate the above discussion with two concrete examples in  $\mathcal{O}_2$ . In these examples, along with the main algebra  $C^*(S_1, S_2) \cong \mathcal{O}_2$ , we consider its other subalgebras, also isomorphic to  $\mathcal{O}_2$ . For example, if  $T_1, T_2$  are isometries in  $C^*(S_1, S_2)$  generating a copy of  $\mathcal{O}_2$ , then we use subscript  $T$  along with the standard notation to indicate that the object comes from  $C^*(T_1, T_2)$  and its generators. Thus  $\varphi_T$  denotes the usual shift on  $C^*(T_1, T_2)$ , that is a map  $\varphi : C^*(T_1, T_2) \rightarrow C^*(T_1, T_2)$  such that  $\varphi(x) = T_1 x T_1^* + T_2 x T_2^*$ . Similarly,  $\mathcal{D}_T$  denotes the diagonal MASA of  $C^*(T_1, T_2)$ , and so on. The proof of one technical lemma needed in Example 3.6 is given afterwards.

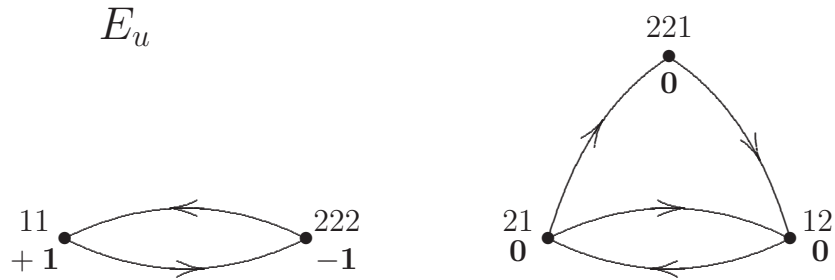
**Example 3.5.** Take  $q_1 = P_{11}$ ,  $q_2 = P_{22}$ , and set

$$A_1 = S_{2221} S_{111}^* + S_{2222} S_{211}^*$$

$$A_2 = S_{111} S_{1222}^* + S_{112} S_{2222}^*$$

$$A_3 = S_{1222} S_{2221}^* + S_{211} S_{112}^* + P_{121} + P_{1221} + P_{212} + P_{221}$$

We note that unitary  $u := A_1 + A_2 + A_3$  falls within the class of polynomial unitaries considered in [4, Section 5]. In particular, its graph  $E_u$ , as defined therein, admits the  $\{-1, 0, +1\}$  labelling:



This labelled graph satisfies the path condition defined in [4, p. 616], and this is an alternative way of showing that  $\lambda_u(\mathcal{F}_2) \subseteq \mathcal{F}_2$ .

Now, we have  $P_{11}\mathcal{O}_2P_{11} \cong \mathcal{O}_2 = C^*(T_1, T_2)$ , under the isomorphism sending  $T_1$  to  $S_{111}S_{11}^*$  and  $T_2$  to  $S_{112}S_{11}^*$ . Similarly,  $P_{222}\mathcal{O}_2P_{222} \cong \mathcal{O}_2 = C^*(R_1, R_2)$ , under the isomorphism sending  $R_1$  to  $S_{2221}S_{222}^*$  and  $R_2$  to  $S_{2222}S_{222}^*$ . Then an easy calculation shows that

$$\text{Ad } u \circ \varphi(T_j) = \varphi_R(R_j),$$

$$\text{Ad } u \circ \varphi(R_j) = \varphi_T(T_j),$$

for  $j = 1, 2$ . Consequently, the restriction of  $(\text{Ad } u \circ \varphi)^2$  to  $P_{11}\mathcal{O}_2P_{11}$  is conjugate to  $\varphi_R^2$ . Likewise, the restriction of  $(\text{Ad } u \circ \varphi)^2$  to  $P_{222}\mathcal{O}_2P_{222}$  is conjugate to  $\varphi_T^2$ . This immediately implies

$$\begin{aligned} \lambda_u(\mathcal{F}_2)' \cap P_{11}\mathcal{O}_2P_{11} &\subseteq \bigcap_{k=1}^{\infty} (\text{Ad } u \circ \varphi)^{2k}(P_{11}\mathcal{O}_2P_{11}) = \mathbb{C}P_{11}, \\ \lambda_u(\mathcal{F}_2)' \cap P_{222}\mathcal{O}_2P_{222} &\subseteq \bigcap_{k=1}^{\infty} (\text{Ad } u \circ \varphi)^{2k}(P_{222}\mathcal{O}_2P_{222}) = \mathbb{C}P_{222}. \end{aligned}$$

That is, both  $P_{11}$  and  $P_{222}$  are minimal projections in  $\lambda_u(\mathcal{F}_2)' \cap \mathcal{O}_2$ . One easily checks that  $\text{Ad } u \circ \varphi(S_{111}S_{222}^*) = S_{222}S_{11}^*$ . Thus  $S_{111}S_{222}^*$  is in  $\lambda_u(\mathcal{F}_2)' \cap \mathcal{O}_2$ , and we see that  $(P_{11} + P_{222})\lambda_u(\mathcal{F}_2)' \cap \mathcal{O}_2(P_{11} + P_{222}) \cong \mathbb{M}_2(\mathbb{C})$ . We remark that the restriction of  $\text{Ad } u \circ \varphi$  to  $(P_{11} + P_{222})\mathcal{O}_2(P_{11} + P_{222})$  is conjugate to endomorphism  $\rho_{1342}$  from [5]. Let

$$w := S_{111}S_{222}^* + S_{222}S_{11}^* + 1 - P_{11} - P_{222}.$$

Then  $w$  is a unitary in  $\lambda_u(\mathcal{F}_2)' \cap \mathcal{O}_2$  such that  $w^*u \in \mathcal{F}_2$ .  $\square$

**Example 3.6.** Take  $q_1 = P_1$ ,  $q_2 = P_{21}$ , and set

$$A_1 = S_{211}S_{21}^* + S_{2121}S_{112}^* + S_{2122}S_{111}^*,$$

$$A_2 = S_{12}S_{121}^* + S_{11}S_{221}^*,$$

$$A_3 = S_{221}S_{122}^* + P_{222}.$$

We put  $u := A_1 + A_2 + A_3$ . By construction,  $\text{Ad } u \circ \varphi(P_1) = P_{21}$  and also  $\text{Ad } u \circ \varphi(P_{21}) = P_1$ . Hence  $\text{Ad } u \circ \varphi(P_{22}) = P_{22}$  as well.

We have  $P_{22}C^*(S_1, S_2)P_{22} \cong \mathcal{O}_2 = C^*(R_1, R_2)$ , under the identification of  $S_{221}S_{22}^*$  with  $R_1$  and  $S_{222}S_{22}^*$  with  $R_2$ . This isomorphism yields a conjugation between the restriction of  $\text{Ad } u \circ \varphi$  to  $P_{22}C^*(S_1, S_2)P_{22}$  and the shift  $\varphi_R$ . Consequently,

$$\lambda_u(\mathcal{F}_2)' \cap P_{22}C^*(S_1, S_2)P_{22} = \bigcap_{k=1}^{\infty} (\text{Ad } u \circ \varphi)^k(P_{22}C^*(S_1, S_2)P_{22}) = \mathbb{C}P_{22}.$$

We have  $P_1C^*(S_1, S_2)P_1 \cong \mathcal{O}_2 = C^*(T_1, T_2)$ , under the identification of  $S_{11}S_1^*$  with  $T_1$  and  $S_{12}S_1^*$  with  $T_2$ . This isomorphism carries the restriction of  $(\text{Ad } u \circ \varphi)^2$  to  $P_1C^*(S_1, S_2)P_1$  to the endomorphism of  $C^*(T_1, T_2)$  given as composition  $\varphi_T \circ \psi_T$ , where  $\psi_T$  is an endomorphism of  $C^*(T_1, T_2)$  such that

$$\psi_T(x) = T_1xT_1^* + T_2(\text{Ad } F_T(x))T_2^*,$$

where  $F_T := T_2 T_1^* + T_1 T_2^*$ . By Lemma 3.7, we have

$$\lambda_u(\mathcal{F}_2)' \cap P_1 C^*(S_1, S_2) P_1 \subseteq \bigcap_{k=1}^{\infty} (\text{Ad } u \circ \varphi)^{2k}(P_1 C^*(S_1, S_2) P_1) = \mathbb{C} P_1.$$

We have  $P_{21} C^*(S_1, S_2) P_{21} \cong \mathcal{O}_2 = C^*(V_1, V_2)$ , under the identification of  $S_{211} S_{21}^*$  with  $V_1$  and  $S_{212} S_{21}^*$  with  $V_2$ . This isomorphism carries the restriction of  $(\text{Ad } u \circ \varphi)^2$  to  $P_{21} C^*(S_1, S_2) P_{21}$  to  $\psi_V \circ \varphi_V$ . An argument similar to that from Lemma 3.7 shows that  $\lambda_u(\mathcal{F}_2)' \cap P_{21} C^*(S_1, S_2) P_{21} = \mathbb{C} P_{21}$ . Alternatively, this also easily follows from the preceding argument and the fact that  $\text{Ad } u \circ \varphi(P_{21}) = P_1$ .

In view of the above, either  $\lambda_u(\mathcal{F}_2)' \cap \mathcal{O}_2 = \text{span}\{P_1, P_{21}, P_{22}\} \cong \mathbb{C}^3$ , or  $P_1$  and  $P_{21}$  are equivalent in  $\lambda_u(\mathcal{F}_2)' \cap \mathcal{O}_2$ . In the latter case,  $\lambda_u(\mathcal{F}_2)' \cap \mathcal{O}_2$  contains a subalgebra isomorphic to  $M_2(\mathbb{C})$  which is invariant under  $\text{Ad } u \circ \varphi$  and has  $P_1$  and  $P_{21}$  as its minimal projections. Suppose for a moment that this is the case. Then  $\text{Ad } u \circ \varphi$  restricts to a non-trivial automorphism of  $M_2(\mathbb{C})$ , by necessity inner. The implementing unitary matrix  $g$  is fixed by  $\text{Ad } u \circ \varphi$  and thus belongs to  $\lambda_u(\mathcal{O}_2)' \cap \mathcal{O}_2$ . Matrix  $g$  has both diagonal entries equal to 0. Multiplying  $g$  by a suitable scalar of modulus 1, we can find such  $g$  that is self-adjoint. Now we see that there is a unitary element  $x$  of  $\mathcal{O}_2$  such that

$$g = S_{21} x^* S_1^* + S_1 x S_{21}^* \in \lambda_u(\mathcal{O}_2)' \cap \mathcal{O}_2.$$

Now, writing  $F := S_1 S_2^* + S_2 S_1^*$ , we compute

$$\begin{aligned} \text{Ad } u \circ \varphi(g) &= u(S_{11} x S_{121}^* + S_{121} x^* S_{11}^* + S_{21} x S_{221}^* + S_{221} x^* S_{21}^*) u^* \\ &= S_{212} F x S_{12}^* + S_{12} x^* F S_{212}^* + S_{211} x S_{11}^* + S_{11} x^* S_{211}^*, \end{aligned}$$

and hence we get

$$S_1 x S_{21}^* + S_{21} x^* S_1^* = S_{212} F x S_{12}^* + S_{12} x^* F S_{212}^* + S_{211} x S_{11}^* + S_{11} x^* S_{211}^*.$$

Multiplying by  $S_1^*$  from the left-side and by  $S_{21}$  from the right-side, we obtain

$$(8) \quad x = S_2 x^* F S_2^* + S_1 x^* S_1^*.$$

Equation (8) implies  $x S_1 = S_1 x^*$  and  $S_1^* x = x^* S_1^*$ . These two combined then yield  $(x + x^*) S_1 = S_1 (x + x^*)$  and  $(x - x^*) S_1 = -S_1 (x - x^*)$ . By [14, Proposition 4], both  $x + x^*$  and  $x - x^*$  are scalars, and thus so is  $x$ . This however contradicts (8).

Thus  $\lambda_u(\mathcal{F}_2)' \cap \mathcal{O}_2 = \text{span}\{P_1, P_{21}, P_{22}\}$  and since  $\tau(P_1) \neq \tau(P_{21})$ , we conclude from Corollary 3.4 that there are no unitaries  $w \in \lambda_u(\mathcal{F}_2)' \cap \mathcal{O}_2$  and  $v \in \mathcal{F}_2$  such that  $u = wv$ .

□

**Lemma 3.7.** *Let  $\psi_T$  be an endomorphism of  $C^*(T_1, T_2) \cong \mathcal{O}_2$  such that*

$$\psi_T(x) = T_1 x T_1^* + T_2 (\text{Ad } F_T(x)) T_2^*,$$

where  $F_T := T_2 T_1^* + T_1 T_2^*$ . Then we have

$$\bigcap_{k=1}^{\infty} (\varphi_T \psi_T)^k (C^*(T_1, T_2)) = \mathbb{C}1.$$

*Proof.* We note that

$$\varphi_T \psi_T(x) = T_{11} x T_{11}^* + T_{21} x T_{21}^* + T_{12} (\text{Ad } F_T(x)) T_{12}^* + T_{22} (\text{Ad } F_T(x)) T_{22}^*.$$

Also, we clearly have  $F_T T_1 = T_2$  and  $F_T T_2 = T_1$ . Thus  $(\varphi_T \psi_T)^k(x)$  may be written as a finite sum of elements of the form  $T_\mu X T_\mu^*$  with  $|\mu| = 2k$ . This gives

$$\bigcap_{k=1}^{\infty} (\varphi_T \psi_T)^k (C^*(T_1, T_2)) \subseteq \mathcal{D}'_T \cap C^*(T_1, T_2) = \mathcal{D}_T.$$

For a positive integer  $k$ , let

$$Q_k := \sum_{|\mu|=k-1} T_{\mu 1} T_{\mu 1}^*.$$

Then a straightforward induction on  $k$  shows that

$$(9) \quad Q_{2k} (\varphi_T \psi_T)^k(x) = Q_{2k} \varphi_T^{2k}(x)$$

for all  $x \in C^*(T_1, T_2)$ . Take a  $d = d^* \in \mathcal{D}_T$  that belongs to  $\bigcap_{k=1}^{\infty} (\varphi_T \psi_T)^k (C^*(T_1, T_2))$ . Suppose  $d$  is not a constant multiple of 1. Then there exist  $k \in \mathbb{N}$ ,  $t \in \mathbb{R}$ ,  $\epsilon > 0$  and  $\mu, \nu \in W_2^{2k-1}$  such that

$$T_{\mu 1} T_{\mu 1}^* d \geq (t + \epsilon) T_{\mu 1} T_{\mu 1}^* \quad \text{and} \quad T_{\nu 1} T_{\nu 1}^* d \leq (t - \epsilon) T_{\nu 1} T_{\nu 1}^*.$$

Let  $x = x^* \in \mathcal{D}_2$  be such that  $d = (\varphi_T \psi_T)^k(x)$ . Then  $Q_{2k} d = Q_{2k} \varphi_T^{2k}(x)$ . Since  $T_{\mu 1} T_{\mu 1}^* \leq Q_{2k}$  and  $T_{\nu 1} T_{\nu 1}^* \leq Q_{2k}$ , we get

$$\begin{aligned} T_{\mu 1} x T_{\mu 1}^* &= T_{\mu 1} T_{\mu 1}^* Q_{2k} \varphi_T^{2k}(x) \geq (t + \epsilon) T_{\mu 1} T_{\mu 1}^*, \quad \text{and} \\ T_{\nu 1} x T_{\nu 1}^* &= T_{\nu 1} T_{\nu 1}^* Q_{2k} \varphi_T^{2k}(x) \leq (t - \epsilon) T_{\nu 1} T_{\nu 1}^*. \end{aligned}$$

This, however, is a contradiction. Indeed, since  $T_{\mu 1}$  and  $T_{\nu 1}$  are isometries, the above two inequalities would imply that both  $x \geq (t + \epsilon)$  and  $x \leq (t - \epsilon)$ . Consequently,

$$\bigcap_{k=1}^{\infty} (\varphi_T \psi_T)^k (C^*(T_1, T_2)) = \mathbb{C}1,$$

as required. □

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NAGOYA INSTITUTE OF TECHNOLOGY, GOKISO-CHO, SHOWA-KU, NAGOYA, AICHI, 466–8555, JAPAN

*E-mail address:* hayashi.tomohiro@nitech.ac.jp

DEPARTMENT OF DATA INFORMATION, KOREA MARITIME AND OCEAN UNIVERSITY, BUSAN 606–791, SOUTH KOREA

*E-mail address:* hongjh@hhu.ac.kr

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, THE UNIVERSITY OF SOUTHERN DENMARK, CAMPUSVEJ 55, DK–5230 ODENSE M, DENMARK

*E-mail address:* szymanski@imada.sdu.dk